Math 409 midterm 2

Name: ____________________

This exam has 7 questions, for a total of 100 points + 10 bonus points.

Please answer each question in the space provided. No aids are permitted.

Question 1. (18 pts)

In each of the following eight cases, indicate whether the given statement is true or false. No justification is necessary.

(a) A sequence $\{x_n\}$ converges if and only if $\{x_n\}$ is bounded.

Solution: False.

(b) $\lim_{x \to 0} \dfrac{e^x + 1}{\sin x + \cos x}$ exists.

Solution: True.

(c) If I is a closed and bounded interval and f is continuous function on I, then there exists $x_0 \in I$ such that $4 - f^2(x_0)$ is the minimum of the function $4 - f^2$ on I.

Solution: True.

(d) Let f and g be two functions defined on $[0,2]$. Suppose g is continuous and $\lim_{x\to 1} f(x) = 0$, then $\lim_{x\to 1} g(x)f(x) = 0$.

Solution: True.

(e) If $f\colon (-1,1) \to \mathbb{R}$ is continuous, then f is bounded on $(-1,1)$.

Solution: False.

(f) Suppose $f\colon (0,1) \to \mathbb{R}$ is uniformly continuous. If $\{x_n\}$ is a Cauchy sequence in $(0,1)$, then $\{f(x_n)\}$ is a Cauchy sequence.

Solution: True.

Question 2. (12 pts)

Prove the function

$$f(x) = \begin{cases} \dfrac{x^2 + x - 2}{x - 1} & x \neq 1 \\ \\ 3 & x = 1 \end{cases}$$

is continuous on $[0, 2]$.

Solution: When $x \neq 1$,

$$f(x) = \frac{x^2 + x - 2}{x - 1}$$

is clearly continuous. When $x = 3$, then

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = 3 = f(1).$$

It follows that f is continuous at $x = 1$. Therefore, f is continuous on $[0, 2]$.

Question 3. (25 pts)

(a) State the Bolzano-Weierstrass theorem.

Solution: Omitted. You can find it in the textbook.

(b) Suppose

$$x_n = \frac{(n^2-1)\sin(n^4+n+1)}{n^2+1}.$$

Show that $\{x_n\}$ has a convergent subsequence.

Solution: Since $|\sin(x)| \leq 1$ for all x, we have

$$|x_n| \leq \frac{n^2-1}{n^2+1} \leq 1.$$

It follows that $\{x_n\}$ is bounded. By the Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergent subsequence.

Question 4. (25 pts)

(a) State the Intermediate Value Theorem.

> **Solution:** Omitted. You can find it in the textbook.

(b) Prove that there exists an $x \in \mathbb{R}$ such that $x^5 + x^3 + x + 2 = \cos(x^5)$.

> **Solution:** Consider the function $g(x) = x^5+x^3+x+2-\cos(x^5)$. On the interval $[-1, 1]$, we have
>
> $$g(-1) = -1 - \cos(-1) < 0 \text{ and } g(1) = 5 - \cos(1) > 0.$$
>
> So we have $g(-1) < 0 < g(1)$. It follows from the intermediate value theorem that there exists $x_0 \in [-1, 1]$ such that $g(x_0) = 0$.

Question 5. (20 pts)

(a) State the definition of uniform continuity.

Solution: Omitted. You can find it in the textbook.

(b) Give an example of a bounded function $f\colon (0,2) \to \mathbb{R}$ which is continuous but not uniformly continuous.

Solution: $f(x) = \sin(1/x)$.

(c) Show that $f(x) = x^2$ is not uniformly continuous on $(0, \infty)$.

Solution: Let $\varepsilon_0 = 1$. Then for any $\delta > 0$, there exist $x = \dfrac{1}{\delta}$ and $y = \dfrac{1}{\delta} + \dfrac{\delta}{2}$ so that $|x - y| = \frac{\delta}{2} < \delta$ and

$$|f(x) - f(y)| = |x - y| \cdot |x + y| > \frac{\delta}{2} \cdot \frac{2}{\delta} = 1.$$

This shows that f is not uniformly continuous.

Bonus Question 6. (5 pts)

If $\{x_n\}$ is a Cauchy sequence, prove that $\{x_n^2\}$ is also a Cauchy sequence.

Solution: If $\{x_n\}$ is a Cauchy sequence, then $\{x_n\}$ converges to some finite real number L. It follows that $\{x_n^2\}$ also converges to L^2. So $\{x_n^2\}$ is Cauchy.

Bonus Question 7. (5 pts)

Let f and g be uniformly continuous functions on $\mathbb{R}$. If both f and g are bounded on $\mathbb{R}$, then fg is also uniformly continuous on $\mathbb{R}$.

Solution: Since f and g are bounded over $\mathbb{R}$, there exists $M > 0$ such that

$$|f(x)| < M \text{ and } |g(x)| < M$$

for all $x \in \mathbb{R}$.

Since f is uniformly continuous on $\mathbb{R}$, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|f(x) - f(y)| < \varepsilon/2M$$

for all $x, y \in \mathbb{R}$ with $|x - y| < \delta_1$. Similarly, there exists $\delta_2 > 0$ such that

$$|g(x) - g(y)| < \varepsilon/2M$$

for all $x, y \in \mathbb{R}$ with $|x - y| < \delta_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &\le |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\ &= |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| < \varepsilon. \end{aligned}$$

This shows that fg is uniformly continuous on $\mathbb{R}$.